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Order of Approximation by Linear Combinations of Positive Linear Operators

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Order of uniform approximation is studied for linear combinations due to May and Rathore of Baskakov-type operators and recent methods of Pethe. The order of approximation is estimated in terms of a higher-order modulus of continuity of the function being approximated. © 1985 Academic Press, Inc.

1. INTRODUCTION

Let $\bar{C}[0, \infty)$ denote the set of functions that are continuous and bounded on the nonnegative axis. For $f \in \bar{C}[0, \infty)$ we consider two classes of positive linear operators.

DEFINITION 1.1. Let $(\phi_n)_{n \in \mathbf{N}}$, $\phi_n\colon [0, b] \to R$ $(b > 0)$ be a sequence of functions having the following properties:

(i) ϕ_n is infinitely differentiable on $[0,b]$;

(ii) $\phi_n(0) = 1$;

(iii) ϕ_n is completely monotone on $[0, b]$, i.e., $(-1)^k \phi_n^{(k)}(x) \geqslant 0$ for $x \in [0, b]$ and $k \in \mathbf{N}_0$;

(iv) there exists an integer c such that

$$-\phi_n^{(k)}(x) = n\phi_{n+c}^{(k-1)}(x)$$

for $x \in [0, b]$, $k \in \mathbf{N}$, $n \in \mathbf{N}$, and $n > \max(c, 0)$.

For $f \in \bar{C}[0, \infty)$, $x \in [0, b]$, and $n \in \mathbf{N}$, define

$$T_n(f; x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \phi_n^{(k)}(x)\, x^k f\left(\frac{k}{n}\right). \tag{1.1}$$

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The positive operators (1.1) specialize well-known methods of Baskakov [1] and Schurer [9]. Recently Lehnhoff [5] has studied uniform approximation properties of (1.1).

DEFINITION 1.2. Let $\theta(y)=\sum_{k=0}^{\infty} a_k y^k$, $|y|<r$, with $a_0=1$. Assume $\theta'(y)=(\theta(y))^p$, $|y|<r$, where $p=1-1/m$, $m\in\mathbf{N}$, or $p\geqslant 1$. Let

$$\theta_n(y)=\sum_{k=0}^{\infty} a_{nk}\,y^k=(\theta(y))^n, \qquad |y|<r.$$

Let [7] $y=g(x)$ be the unique solution to the equation

$$\frac{y\theta'(y)}{\theta(y)}=y(\theta(y))^{p-1}=x$$

with $g(0)=0$. There exists [7] $b\in(0,r)$ such that $g(x)>0$ for $0<x\leqslant b$. For $f\in\bar{C}[0,\infty)$, $x\in[0,b]$, and $n\in\mathbf{N}$, define

$$S_n(f;x)=\frac{1}{\theta_n(g(x))}\sum_{k=0}^{\infty} a_{nk}(g(x))^k f\left(\frac{k}{n}\right). \tag{1.2}$$

The methods (1.2) specialize ones introduced by S. Pethe [7], who showed uniform convergence of (1.2) on $[0,b]$. Since $p=1-1/m$, $m\in\mathbf{N}$, or $p\geqslant 1$, it follows that $a_{nk}\geqslant 0$ and S_n is a positive linear operator. Pethe notes that the methods of Bernstein, Baskakov, and Szasz are obtained with $\Theta(y)=1+y$ $(p=0)$, $\theta(y)=(1-y)^{-1}$ $(p=2)$, and $\theta(y)=e^y$ $(p=1)$, respectively.

May [6] and Rathore [8] have described a method for forming linear combinations of positive linear operators, so as to improve the order of approximation. We apply this technique to (1.1) and (1.2).

Let $f\in\bar{C}[0,\infty)$, $x\in[0,b]$, $k\in\mathbf{N}_0$, and $P_n(f;x)$ denote either (1.1) or (1.2). The linear combination is given by

$$L_n(f;k;x)=\sum_{j=0}^{k} c(j,k)\,P_{d_jn}(f;x), \tag{1.3}$$

where $d_0, d_1,..., k_k$ are $k+1$ arbitrary, fixed, and distinct positive integers and

$$c(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i}, \qquad k>0 \qquad \text{and} \qquad c(0,0)=1.$$

Let $\|\cdot\|_b$ and $\|\cdot\|_\infty$ denote the norms of spaces $C[0, b]$ and $\bar{C}[0, \infty)$, respectively. For $f \in \bar{C}[0, \infty)$,

$$\omega_m(f; \delta) = \sup_{0 \leqslant t \leqslant \delta} \sup_{0 \leqslant x < \infty} \left| \sum_{v=0}^{m} \binom{m}{v} (-1)^{m-v} f(x+vt) \right|$$

is the modulus of smoothness of order m. In the next section we establish

$$\|L_n(f: k; \cdot) - f\|_b \leqslant M_k [n^{-(k+1)} \|f\|_\infty + \omega_{2k+2}(f; n^{-1/2})]$$

for all n sufficiently large, where M_k is a positive constant that depends on k but is independent of f and n.

2. Order of Approximation

In the sequel $f \in \bar{C}[0, \infty)$, $x \in [0, b]$, and $P_n(f; x)$ denotes either (1.1) or (1.2). For $n \in \mathbf{N}$ and $s \in \mathbf{N}_0$ write

$$M_{n,s}(x) = n^s P_n((t-x)^s; x).$$

Lemma 2.1. *For $m \in \mathbf{N}_0$, $n \in \mathbf{N}$, and $n > \max(c, 0)$ we have the recurrence relation*

$$M_{n,m+1}(x) = nx \sum_{s=0}^{m} \binom{m}{s} (1-cx)^{m-s} M_{n+c,s}(x) - nx M_{n,m}(x).$$

Here $c = 1 - p$ for operator (1.2) *and c is given by Definition* 1.1 *for operator* (1.1).

Proof. The relation for operator (1.1) is due to Sikkema [10].

Assume $P_n(f; x)$ is operator (1.2). Using the notation of Definition 1.2, it is easy to obtain the result

$$n a_{n+p-1,k-1} = k a_{nk}. \tag{2.1}$$

Using (2.1) and Definition 1.2, we have

$$\begin{aligned} M_{n,m+1}(x) &= \sum_{k=0}^{\infty} \frac{a_{nk}(g(x))^k}{[\theta(g(x))]^n} (k-nx)^{m+1} \\ &= g(x) \sum_{k=1}^{\infty} \frac{k a_{nk}(g(x))^{k-1}}{[\theta(g(x))]^n} (k-nx)^m - nx M_{n,m}(x) \\ &= \frac{n g(x)}{[\theta(g(x))]^{1-p}} \sum_{k=1}^{\infty} \frac{a_{n+p-1,k-1}(g(x))^{k-1}}{[\theta(g(x))]^{n+p-1}} [(k-1)-(n+p-1)x \\ &\quad + 1 + (p-1)x]^m - nx M_{n,m}(x) \end{aligned}$$

$$= nx \sum_{k=1}^{\infty} \frac{a_{n+p-1,k-1}(g(x))^{k-1}}{[\theta(g(x))]^{n+p-1}} \sum_{s=0}^{m} \binom{m}{s} [(k-1)$$

$$-(n+p-1)x]^s(1+px-x)^{m-s} - nxM_{n,m}(x)$$

$$= nx \sum_{s=0}^{m} \binom{m}{s} (1+px-x)^{m-s} \sum_{k=1}^{\infty} \frac{a_{n+p-1,k-1}(g(x))^{k-1}}{[\theta(g(x))]^{n+p-1}}$$

$$\times [(k-1)-(n+p-1)x]^s - nxM_{n,m}(x)$$

$$= nx \sum_{s=0}^{m} \binom{m}{s} (1-cx)^{m-s} M_{n-c,s}(x) - nxM_{n,m}(x)$$

for $m \in \mathbf{N}_0$ and $n > \max(c, 0)$. Also,

$$M_{n,0}(x) = 1.$$

The next lemma was proved by Lehnhoff [5] for operator (1.1). Using Lemma 2.1, the proof for oeprator (1.2) is exactly the same.

LEMMA 2.2. *For $m \in \mathbf{N}$, $n \in \mathbf{N}$, and $n > \max(c, 0)$ the formula*

$$M_{n,m}(x) = \sum_{j=0}^{[m/2]} \psi_{m,j}(x)\, n^j$$

holds, where $\psi_{m,j}$ $(0 \leqslant j \leqslant [m/2])$ is an algebraic polynomial of degree m in x. Moreover, there exists a positive constant $\alpha(m, b)$ such that

$$|M_{n,m}(x)| \leqslant \alpha(m, b)\, n^{[m/2]}$$

and

$$|P_n((t-x)^m; x)| \leqslant \alpha(m, b)\, n^{-[(m+1)/2]}$$

hold uniformly for all $x \in [0, b]$.

LEMMA 2.3. *For $x \in [0, b]$, $j \in \mathbf{N}$, $n \in \mathbf{N}$, and $n > \max(c, 0)$,*

$$0 \leqslant P_n((t-x)^{2j}; x) \leqslant \alpha(j, b) n^{-j}.$$

Proof. Use Lemma 2.2 and the fact that P_n is a positive operator. In the sequel $L_n(f; k; x)$ denotes the linear combination (1.3).

LEMMA 2.4. *We have*

$$L_n(1; k; x) = 1$$

and, for $v = 1, 2, \ldots, 2k+1$,

$$\|L_n((t-\cdot)^v; k; \cdot)\|_b = O(n^{-(k+1)}), \qquad n \to \infty.$$

Proof. Using [6, p. 1228],

$$L_n(1; k; x) = \sum_{j=0}^{k} c(j, k)\, P_{d_j n}(1; x) = \sum_{j=0}^{k} c(j, k) = 1.$$

Next, for $v = 1, 2, \ldots, 2k+1$ and n sufficiently large, it follows from Lemma 2.2 that

$$\begin{aligned} L_n((t-x)^v; k; x) &= \sum_{j=0}^{k} c(j, k)\, P_{d_j n}((t-x)^v; x) \\ &= \sum_{j=0}^{k} c(j, k)(d_j n)^{-v} \sum_{s=0}^{[v/2]} \psi_{v,s}(x)(d_j n)^s \\ &= \sum_{s=0}^{[v/2]} \frac{\psi_{v,s}(x)}{n^{v-s}} \sum_{j=0}^{k} c(j, k)\, d_j^{-(v-s)}. \end{aligned}$$

Since

$$\sum_{j=0}^{k} c(j, k)\, d_j^{-(v-s)} = 0$$

for $v - s = 1, 2, \ldots, k$ [6, p. 1228], we have

$$|L_n((t-x)^v; k; x)| \leqslant \frac{1}{n^{k+1}} \sum_{s=0}^{[v/2]} \frac{|\psi_{v,s}(x)|}{n^{(v-s)-(k+1)}} \sum_{j=0}^{k} |c(j, k)|\, d_j^{s-v} \leqslant \beta n^{-(k+1)},$$

where β is a constant that depends on k and b but is independent of n.

The next result follows from the fact that $P_n(1; x) = 1$ for $x \in [0, b]$.

LEMMA 2.5. *For* $f \in \bar{C}[0, \infty)$ *and* $n \in \mathbf{N}$,

$$\|P_n(f)\|_b \leqslant \|f\|_\infty .$$

Lemma 2.5 implies that (1.3) is a uniformly bounded sequence of linear operators from $\bar{C}[0, \infty)$ into $C[0, b]$. Our final lemma extends a result of Freud and Popov [3].

LEMMA 2.6. *For an arbitrry* $f \in \bar{C}[0, \infty)$, *for every* $m \in \mathbf{N}$, *and for every* $\delta \in (0, 1/m)$, *there exists a function* $f_{m,\delta}$ *such that*

$$f_{m,\delta} \in \bar{C}[0, \infty); \tag{2.2}$$

$$f_{m,\delta}^{(m)} \in \bar{C}[0, \infty); \tag{2.3}$$

$$\|f - f_{m,\delta}\|_\infty \leqslant M_m^{(1)} \omega_m(f; \delta); \tag{2.4}$$

$$\|f_{m,\delta}^{(m)}\|_\infty \leqslant M_m^{(2)} \delta^{-m} \omega_m(f; \delta), \tag{2.5}$$

where $M_m^{(1)}$, $M_m^{(2)}$ are positive costants depending only on m.

Proof. For $f \in \bar{C}[0, \infty)$, $m \in \mathbf{N}$, $\delta \in (0, 1/m)$, and $t \geqslant 0$, define [3, p. 170]

$$f_{m,\delta}(t) = \frac{1}{\delta^m} \left(\int_0^\delta\right)^m \sum_{v=1}^{m} \binom{m}{v} (-1)^{m-v} f\left[t + \frac{v}{m}\left(t_1 + \cdots + t_m\right)\right] dt_1 \cdots dt_m.$$

Since $f \in \bar{C}[0, \infty)$, (2.2) follows easily. Results (2.3), (2.4), and (2.5) follow from calculations of Freud and Popov [3, pp. 170, 171].

THEOREM 2.7. *If $f \in \bar{C}[0, \infty)$ then, for all n sufficiently large,*

$$\|L_n(f; k; \cdot) - f\|_b \leqslant M_k[n^{-(k+1)}\|f\|_\infty + \omega_{2k+2}(f; n^{-1/2})],$$

where M_k is a positive constant that depends on k but is independent of f and n.

Proof. For $f \in \bar{C}[0, \infty)$ and $k \in \mathbf{N}_0$ let $f_{2k+2,\delta}$ be given by Lemma 2.6. Since $f_{2k+2,\delta}^{(2k+2)} \in C[0, \infty)$, we can write, for $x \in [0, b]$ and $t \geqslant 0$,

$$\begin{aligned} f_{2k+2,\delta}(t) = f_{2k+2,\delta}(x) &+ \sum_{v=1}^{2k+1} \frac{f_{2k+2,\delta}^{(v)}(x)}{v!} (t - x)^v \\ &+ \frac{f_{2k+2,\delta}^{(2k+2)}(\xi(t))}{(2k+2)!} (t - x)^{2k+2}. \end{aligned} \tag{2.6}$$

It follows easily from (2.6), [4, p. 5], Lemma 2.3, and Lemma 2.4 that

$$\|L_n(f_{2k+2,\delta}; k; \cdot) - f_{2k+2,\delta}\|_b \leqslant \gamma_k(\|f_{2k+2,\delta}\|_\infty + \|f_{2k+2,\delta}^{(2k+2)}\|_\infty) n^{-(k+1)}, \tag{2.7}$$

for all n sufficiently large, where γ_k is a constant that depends on k but is independent of n.

Let $f \in \bar{C}[0, \infty)$ and write

$$\begin{aligned} L_n(f; k; x) - f(x) = L_n(f - f_{2k+2,\delta}; k; x) &+ L_n(f_{2k+2,\delta}; k; x) \\ &- f_{2k+2,\delta}(x) + f_{2k+2,\delta}(x) - f(x). \end{aligned} \tag{2.8}$$

Choose $\delta = n^{-1/2}$ and Theorem 2.7 follows from (2.7), (2.8), Lemma 2.6, and the remark following Lemma 2.5.

The following example shows the estimate of Theorem 2.7 is best possible for linear combinations (1.3) of either (1.1) or (1.2).

EXAMPLE 2.8. Let $0 < x_0 < 1$, $0 < \alpha \leqslant 1$, and

$$\begin{aligned} f(x) &= |x - x_0|^\alpha, && 0 \leqslant x \leqslant 1, \\ &= f(1), && x > 1. \end{aligned}$$

Choose $\phi_n(x) = (1-x)^n$, $0 \leqslant x \leqslant 1$, in Definition 1.1 so that (1.1) becomes $B_n(f; x)$, the nth Bernstein polynomial, and choose $\theta(y) = 1 + y$ in Definition 1.2 so that (1.2) also becomes $B_n(f; x)$. Form the linear combination

$$L_n(f; k; x) = \sum_{j=0}^{k} c(j, k)\, B_{2^j n}(f; x)$$

for $k \geqslant 1$ and $0 \leqslant x \leqslant 1$, where $c(j, k)$ are as in (1.3). This is a linear combination due to Butzer [2, 6]. Let $\|\cdot\|$ denote the sup norm on $C[0, 1]$. We have

$$\|f - L_n(f; k; \cdot)\| \geqslant |f(x_0) - L_n(f; k; x_0)| \geqslant A_k n^{-\alpha/2}, \tag{2.9}$$

where A_k is a positive constant that depends on k. Estimate (2.9) was shown by Butzer [2] for $k = 1$ and, as he pointed out, the same method of proof can be applied for $k > 1$. Next, Theorem 2.7 yields

$$\|f - L_n(f; k; \cdot)\| \leqslant B_k n^{-\alpha/2},$$

where B_k is a positive constant that depends on k.

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